Critical exponents to order ϵ^3 for ϕ^3 models of critical phenomena in 6-ϵ dimensions

1980 J. Phys. A: Math. Gen. 13 L247

(http://iopscience.iop.org/0305-4470/13/7/006)

J. Phys. A: Math. Gen. **13** (1980) L247–L251. Printed in Great Britain

LETTER TO THE EDITOR

Critical exponents to order ϵ^3 for ϕ^3 models of critical phenomena in $6-\epsilon$ dimensions

O F de Alcantara Bonfim†, J E Kirkham‡ and A J McKane§

† Department of Physics, University of Edinburgh, James Clerk Maxwell Building, Mayfield Road, Edinburgh EH9 3JZ, UK

‡ Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

§ Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

Received 18 April 1980

Abstract. We study scalar field theories for which the interaction term of the Hamiltonian is cubic in the fields. We indicate the circumstances for which field theory models of this type represent continuous phase transitions. The renormalisation group functions for these models are presented up to, and including, three-loop contributions, giving critical exponents to order ϵ^3 in $6-\epsilon$ dimensions. The exponent σ which characterises the Yang–Lee edge singularity is given explicitly to this order.

The field theoretical approach to the study of critical phenomena has been extremely successful in giving a unified picture of continuous phase transitions, as well as predicting physical quantities in the critical region (see for instance Amit 1978). In particular, the idea that for a given system there exists a critical dimension, d_c, above which mean field theory is exact, led to the ϵ-expansion of Wilson and Fisher (1972). In this scheme the corrections to mean field theory for dimensionality $d < d_c$ are calculated perturbatively in $\epsilon = d_c - d$. The application of this method to the field theory with quartic interaction $(\boldsymbol{\phi}^2)^2$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ has led to an expansion in $\epsilon = 4-d$ to fourth order for the critical exponents of the Ising and Heisenberg systems (Kazakov *et al* 1979). If these series are resummed taking into account the high-order behaviour of the theory then, for $d = 3$, they are found to be in impressive agreement with experimental and high-temperature series results. For these systems cubic interactions are forbidden by symmetry requirements; however, in general such interactions may be present. Furthermore, naive dimensional analysis shows us that near six dimensions (the critical dimension for cubic interactions) the cubic interaction dominates those of higher orders (Amit *et al* 1977). Thus for cubic interactions we can calculate critical exponents as expansions in $\epsilon = 6-d$. The results to second order were evaluated by Amit (1976) and Priest and Lubensky (1976); however these results give insufficient information for reliable numerical results to be obtained. In an attempt to remedy this situation we have calculated the third-order contributions.

We consider models which have the Hamiltonian

$$H = \int \mathrm{d}^d x [\tfrac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \tfrac{1}{2}m_0^2\phi_i\phi_i + (g_0/3!)d_{ijk}\phi_i\phi_j\phi_k] \qquad (1)$$

where $i, j, k = 1, \ldots, n$, repeated indices are summed and d_{ijk} is an invariant third-rank symmetric tensor of some symmetry group. The critical exponents are calculated from the renormalisation group functions at the fixed point of the theory. The exponents are related to the anomalous dimensions of ϕ and ϕ^2($\gamma_\phi(g)$ and $\gamma_{\phi^2}(g)$ respectively) by the relations

$$\eta = \gamma_\phi(g^*), \tag{2}$$

$$\nu^{-1} - 2 + \eta = \gamma_{\phi^2}(g^*), \tag{3}$$

where g^* is the fixed point of the theory (Amit 1978). The other exponents are given by scaling laws.

Hamiltonians with a cubic interaction are used to model many phase transitions. Examples are the isotropic to nematic phase transition in liquid crystals (de Gennes 1969, Priest and Lubensky 1976), any system described by the Potts model (Potts 1952, Zia and Wallace 1975, Priest and Lubensky 1976), in particular the percolation problem (Harris *et al* 1975, Amit 1976, Priest and Lubensky 1976) which is the $n \to 0$ limit of the $(n+1)$ state Potts model (Fortuin and Kasteleyn 1972), the Yang–Lee edge singularity (Fisher 1978), the Edwards–Anderson model of a spin glass (Edwards and Anderson 1975, Harris *et al* 1976), as well as quantum field theory models in particle physics (McKane *et al* 1976, Wallace 1979).

We wish to divide these models into two classes: those with real Hamiltonians and in which no 'unphysical' limits are taken and those where an 'unphysical' limit is taken. In the second class we place the percolation problem and the Edwards–Anderson model (in which the limit $n \to 0$ is taken) and Fisher's formulation of the Yang–Lee edge singularity (in which the coupling constant is pure imaginary).

Two important remarks should be made. Firstly, field theories in the first class have instanton solutions (McKane 1979), indicating the ground state instability of such models, from which it follows that the ϵ-expansion is not well defined: not only does it diverge but it has non-oscillatory behaviour at large orders in ϵ. This is not true of the percolation problem (Houghton *et al* 1978) nor of the Yang–Lee edge problem (Kirkham and Wallace 1979). Secondly, although by naive dimensional analysis quartic and higher interactions are irrelevant in the critical region, this situation may change after renormalisation and the irrelevance of these operators should be checked *a posteriori* (Amit 1978). For many of the theories in the first class, it is found (Wallace 1979) that even if a fixed point exists it becomes unstable to ϕ^4 operators as d is lowered from six, whereas ϕ^4 operators are irrelevant in cases of interest in the second class (Amit *et al* 1977, Elderfield and McKane 1978, Kirkham and Wallace 1979). Below, we give our results for the renormalisation group functions and critical exponents starting from the general Hamiltonian (1), although the above discussion makes it clear that we will be largely interested in models falling into the second class.

The different models are specified in (1) solely by the tensor d_{ijk}. The Feynman diagrams to be evaluated are those of a one-component ϕ^3 theory, but multiplied by the appropriate tensorial contraction. In the one-loop calculation only two types of contraction are found, defining two constants α and β by

$$d_{ikl}d_{jkl} = \alpha\delta_{ij}, \qquad d_{ilm}d_{jmn}d_{knl} = \beta d_{ijk}. \tag{4}$$

These correspond to the diagrams in figure 1(a) and figure 1(b) respectively, with the tensors situated at each vertex, repeated indices being represented by internal lines and free indices by external lines. At two and three loops, three more constants are

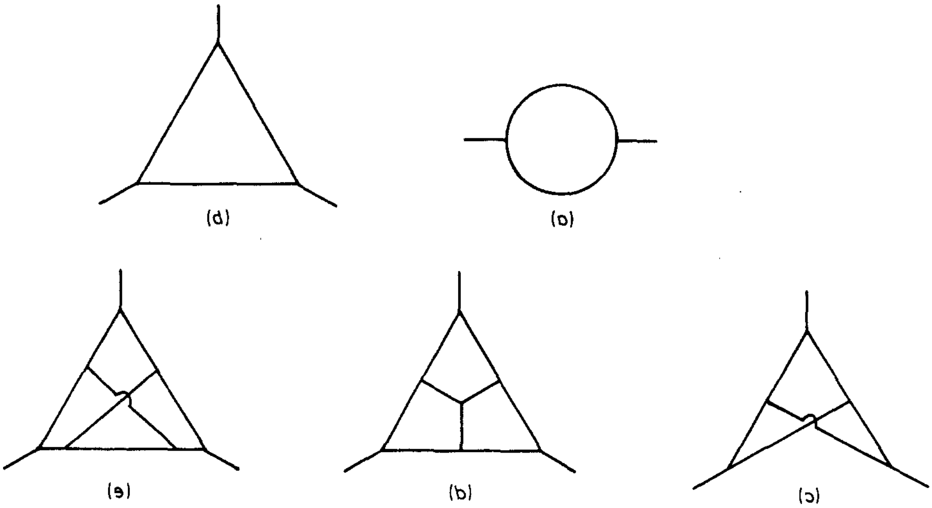

Figure 1. The five independent types of tensor contractions at three loops.

encountered: γ, δ and λ given diagrammatically in figures 1(c), 1(d) and 1(e) respectively. The greatest amount of labour involved in the calculations of the renormalisation group functions is in the evaluation of the Feynman graph integrals. This will be discussed elsewhere: here we simply give our results. They are

$$\beta(g) = -\tfrac{1}{2}\epsilon g + (\tfrac{1}{4}\alpha - \beta)g^3 + (-\tfrac{11}{144}\alpha^2 + \tfrac{11}{24}\alpha\beta - \tfrac{3}{4}\beta^2 - \tfrac{1}{2}\gamma)g^5$$
$$+ [\tfrac{821}{20736}\alpha^3 - \tfrac{1013}{3456}\alpha^2\beta + \tfrac{349}{576}\alpha\beta^2 + \tfrac{9}{16}\beta^3 + \alpha\gamma(\tfrac{43}{96} - \tfrac{1}{8}\zeta(3))$$
$$+ \beta\gamma(\tfrac{5}{2}\zeta(3) - \tfrac{71}{24}) - \delta + \lambda(1 - 3\zeta(3))]g^7 + \mathrm{O}(g^9), \tag{5}$$

$$\gamma_\phi(g) = \tfrac{1}{6}\alpha g^2 + (\tfrac{1}{9}\alpha\beta - \tfrac{11}{216}\alpha^2)g^4 + [\tfrac{821}{31104}\alpha^3 - \tfrac{179}{1728}\alpha^2\beta + \tfrac{85}{864}\alpha\beta^2$$
$$+ \alpha\gamma(\tfrac{7}{48} - \tfrac{1}{12}\zeta(3))]g^6 + \mathrm{O}(g^8), \tag{6}$$

$$\gamma_{\phi^2}(g) = \alpha g^2 + (\alpha\beta - \tfrac{1}{24}\alpha^2)g^4 + [\tfrac{95}{216}\alpha^3 + \alpha^2\beta(-\tfrac{79}{96} - \tfrac{1}{2}\zeta(3))$$
$$+ \alpha\beta^2(\tfrac{65}{48} + \zeta(3)) + \tfrac{7}{8}\alpha\gamma]g^6 + \mathrm{O}(g^8), \tag{7}$$

where $\epsilon = 6 - d$, $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1{\cdot}202\ldots$ and where, as usual, a factor of $S_d/(2\pi)^d$ (S_d is the surface area of a unit sphere in d dimensions) has been absorbed into g^2. The two-loop results agree with those appearing in the literature, in the cases where the same renormalisation scheme was used (MacFarlane and Woo 1974, Amit 1976). From equation (5) we find that the non-trivial fixed point is given by

$$g^{*2} = \frac{2\epsilon}{(\alpha - 4\beta)} + \frac{\epsilon^2}{(\alpha - 4\beta)^3}\left(\frac{11}{9}\alpha^2 - \frac{22}{3}\alpha\beta + 12\beta^2 + 8\gamma\right)$$
$$+ \frac{\epsilon^3}{(\alpha - 4\beta)^5}\left(\frac{49}{216}\alpha^4 - \frac{1127}{324}\alpha^3\beta + \frac{1415}{54}\alpha^2\beta^2 - \frac{1048}{9}\alpha\beta^3 + 216\beta^4\right.$$
$$+ \alpha^2\gamma\left(\frac{47}{9} + 4\zeta(3)\right) + \beta^2\gamma\left(-\frac{560}{3} + 320\zeta(3)\right) + \alpha\beta\gamma\left(\frac{104}{3} - 96\zeta(3)\right)$$
$$\left. + 32\delta(\alpha - 4\beta) + 32\lambda(\alpha - 4\beta)(3\zeta(3) - 1) + 64\gamma^2\right) + \mathrm{O}(\epsilon^4) \tag{8}$$

which, using equations (2), (3), (6) and (7), gives

$$\eta=\frac{\alpha\epsilon}{3(\alpha-4\beta)}+\frac{\alpha\epsilon^2}{(\alpha-4\beta)^3}\left(\frac{\alpha\beta}{27}+\frac{2\beta^2}{9}+\frac{4\gamma}{3}\right)$$
$$+\frac{\alpha\epsilon^3}{(\alpha-4\beta)^5}\Big(-\frac{7}{108}\alpha^3\beta+\frac{637}{486}\alpha^2\beta^2-\frac{292}{27}\alpha\beta^3+\frac{736}{27}\beta^4+\frac{11}{27}\alpha^2\gamma$$
$$+\alpha\beta\gamma(-\tfrac{32}{3}\zeta(3)+\tfrac{176}{27})+\beta^2\gamma(-\tfrac{80}{3}+\tfrac{128}{3}\zeta(3))+\tfrac{32}{3}\gamma^2$$
$$+\tfrac{16}{3}\delta(\alpha-4\beta)+\tfrac{16}{3}\lambda(\alpha-4\beta)(3\zeta(3)-1)\Big)+\mathrm{O}(\epsilon^4) \quad (9)$$

and

$$\nu^{-1}-2+\eta=\frac{2\alpha\epsilon}{(\alpha-4\beta)}+\frac{\alpha\epsilon^2}{(\alpha-4\beta)^3}\left(\frac{19\alpha^2}{18}-\frac{8\alpha\beta}{3}-4\beta^2+8\gamma\right)$$
$$+\frac{\alpha\epsilon^3}{(\alpha-4\beta)^5}\Big[\frac{85}{24}\alpha^4-\alpha^3\beta\Big(\frac{2534}{81}+4\zeta(3)\Big)+\alpha^2\beta^2\Big(\frac{812}{9}+40\zeta(3)\Big)$$
$$-\alpha\beta^3(\tfrac{1216}{9}+128\zeta(3))+\beta^4(\tfrac{592}{3}+128\zeta(3))+\alpha^2\gamma(\tfrac{98}{9}+4\zeta(3))$$
$$+\alpha\beta\gamma(16-96\zeta(3))+\beta^2\gamma(320\zeta(3)-\tfrac{608}{3})+64\gamma^2$$
$$+32\delta(\alpha-4\beta)+32\lambda(\alpha-4\beta)(3\zeta(3)-1)\Big]+\mathrm{O}(\epsilon^4). \quad (10)$$

These agree with the $\mathrm{O}(\epsilon^2)$ results of Priest and Lubensky (1975) and Amit† (1976) and with the $\mathrm{O}(\epsilon^3)$ evaluation of η in the special case of a symmetry for which $\beta=\gamma=\lambda=0$ (McKane 1977).

The simplest case of interest is the one where $n=1$ and g is pure imaginary, this being the model considered by Fisher (1978) in his study of Yang–Lee edge singularities. This effectively means $d_{111}=\mathrm{i}$, leading to $\alpha=\beta=\delta=\lambda=-1$ and $\gamma=1$. The exponent which characterises the singularities is

$$\sigma=(d-2+\eta)/(d+2-\eta)$$
$$=\tfrac{1}{2}-\tfrac{1}{12}\epsilon-\tfrac{79}{3888}\epsilon^2+(\tfrac{1}{81}\zeta(3)-\tfrac{10445}{1259712})\epsilon^3+\mathrm{O}(\epsilon^4). \quad (11)$$

The one-component theory provides a further check on our results since in this case the anomalous dimension $\gamma_{\phi^2}(g)$ is linearly related to $\gamma_\phi(g)$ and $\beta(g)$. (This is proved using the equation of motion of the theory (Amit 1978).) A consequence of this is that the exponent $\beta=\frac{1}{2}\nu\,(d-2+\eta)$ is exactly equal to its mean field value of 1.

In order to obtain a realistic value for σ when $\epsilon=3$, the series (11) needs to be analysed using resummation techniques. This, together with a more detailed description of the calculation and other applications, will be discussed elsewhere.

Acknowledgments

We wish to thank Professor D J Wallace for communicating the work of the Dubna group to us and also for encouraging us to undertake the present calculation. In particular OFAB thanks Professor D J Wallace for invaluable help and CNPq (Brazilian Agency) for financial support.

† There is a misprint in equation (6.23) of Amit's paper; the coefficient of $\alpha_1\beta_1$ should be $-\frac{1}{3}$, not $+\frac{1}{3}$.

References

Amit D J 1976 *J. Phys. A: Math. Gen.* **9** 1441
—— 1978 *Field Theory, the Renormalization Group, and Critical Phenomena* (New York: McGraw-Hill)
Amit D J, Wallace D J and Zia R K P 1977 *Phys. Rev.* B **15** 4657
Edwards S F and Anderson P W 1975 *J. Phys. F: Metal Phys.* **5** 965
Elderfield D J and McKane A J 1978 *Phys. Rev.* B **18** 3730
Fisher M E 1978 *Phys. Rev. Lett.* **40** 1610
Fortuin C M and Kasteleyn P W 1972 *Physica (Utr.)* **57** 536
de Gennes P G 1969 *Phys. Lett.* **30A** 454
Harris A B, Lubensky T C and Chen J-H 1976 *Phys. Rev. Lett.* **36** 415
Harris A B, Lubensky T C, Holcomb W K and Dasgupta C 1975 *Phys. Rev. Lett.* **35** 327 and *Erratum* (1975) **35** 1397
Houghton A, Reeve J S and Wallace D J 1978 *Phys. Rev.* B **17** 2956
Kazakov D I, Tarasov O V and Vladimirov A A 1979 *Dubna Preprint* **E2**-12249
Kirkham J E and Wallace D J 1979 *J. Phys. A: Math. Gen.* **12** L47
MacFarlane A J and Woo G 1974 *Nucl. Phys.* B **77** 91 and *Erratum* (1975) B **86** 548
McKane A J 1977 *J. Phys. G: Nucl. Phys.* **3** 1165
—— 1979 *Nucl. Phys.* B **152** 166
McKane A J, Wallace D J and Zia R K P 1976 *Phys. Lett.* **65B** 171
Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
Priest R G and Lubensky T C 1976, *Phys. Rev.* B **13** 4159 and *Erratum* (1976) B **14** 5125
Wallace D J 1979 in *Hadron Structure and Lepton-Hadron Interactions* (Cargèse 1977) ed M Levy (New York: Plenum) p 687
Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240
Zia R K P and Wallace D J 1975 *J. Phys. A: Math. Gen.* **8** 1495